Turbulence without pressure in d dimensions

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The randomly driven Navier-Stokes equation without pressure in d-dimensional space is considered as a model of strong turbulence in a compressible fluid. We derive a closed equation for the velocity-gradient probability density function. We find the asymptotics of this function for the case of the gradient velocity field (Burgers turbulence) and provide a numerical solution for the two-dimensional case. Application of these results to the velocity-difference probability density function is discussed. [S1063-651X(99)06203-0]

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I. INTRODUCTION

The Burgers equation with a random external force is considered to be the first exactly solvable model of one-dimensional (1D) turbulence and has been extensively studied in recent years [1–11]. Though rather simplified, this model can serve as a test model for some general ideas within the theory of strong turbulence. In 1995, methods of quantum field theory were applied to this problem by Polyakov [1] which enabled a qualitative explanation of velocity-difference probability-density functions (PDF's) measured numerically by Chekhlov and Yakhot [2]. In [3] it was shown that the approach [1] allows one to obtain quantitatively correct results. Extensive numerical simulations published recently by Gotoh and Kraichnan [10] show that the predictions of [1,3] are quite accurate and coincide with numerical simulations to within about 5%. Yakhot has shown in [12] that the ideas introduced in [1] can have much wider application and can also work for incompressible velocity fluctuations.

We believe that the operator product expansion (OPE), introduced in [1] to take into account the viscous term, is an adequate language to treat compressible turbulence in higher dimensions as well, where shock structures and associated local dissipation persist. In the present paper we find a closed equation for the velocity-gradient PDF for compressible turbulence in any number of dimensions. We investigate the asymptotics of the PDF and present the numerical solution for the 2D case.

The basic equation we will study is the following:

$$\mathbf{u}_t+(\mathbf{u}\cdot\boldsymbol{\nabla})u=\nu\,\Delta\mathbf{u}+\mathbf{f}. \tag{1}$$

The force f is chosen to be Gaussian with zero mean and white in time variance,

$$\langle f^i(\mathbf{x},t)f^k(\mathbf{x}',t')\rangle=\delta(t-t')\,\kappa^{ik}(\mathbf{x}-\mathbf{x}'), \tag{2}$$

where the κ function is concentrated at some large scale L and can be expanded as follows:

$$\kappa^{ik}(\mathbf{y})=\kappa_0\delta^{ik}-\kappa_1(y^2\delta^{ik}+2\alpha y^iy^k) \tag{3}$$

for $y\ll L$. We assume that the steady states for velocity gradient and velocity difference exist; for this we can require, for example, that periodic boundary conditions on a scale much larger than L be imposed and that the zero harmonic in the κ function be absent. These assumptions are usually used in numerical simulations [2,10].

In this paper we appeal to the results obtained for the 1D Burgers turbulence without pressure in [1–4,6,10]. In particular, we are interested in the velocity-gradient PDF $P(\partial u^i/\partial x^k)$ and the velocity-difference PDF $P_v(\mathbf{u}(\mathbf{x}_1)-\mathbf{u}(\mathbf{x}_2))$, where the velocities are taken at the same time at some fixed points $\mathbf{x}_1$ and $\mathbf{x}_2$. The physical picture presented in these papers allows us to consider a general phenomenon such as intermittency on a rigorous basis; it is related to the spontaneous breakdown of the Galilean invariance of the forced equation and to the algebraic decay of the PDF's. We will not repeat these arguments here; instead, we will concentrate on the main ideas which allow us to consider the multidimensional case.

We will be interested in the case of small dissipation ν and will consider distances $|x_1-x_2|\ll L$. The following order of the limits should be considered to get the steady state: we first set $t\to\infty$ and then consider the limit $\nu\to 0$.

II. VELOCITY-GRADIENT PDF

To proceed quantitatively, consider the following characteristic function (Z function) for the velocity gradient $u_k^i\equiv\partial u^i/\partial x^k$:

$$Z(\sigma_{kl})\equiv\langle\exp(i\sigma_{kl}u_l^k)\rangle. \tag{4}$$

As a result of Eq. (1), this function satisfies the following Fokker-Planck equation:

$$\begin{aligned}\dot{Z}=&\,i\sigma_{ik}\frac{\partial^2}{\partial\sigma_{lk}\partial\sigma_{il}}Z-i\,\delta_{ik}\frac{\partial}{\partial\sigma_{ik}}Z\\&-\left[\alpha\sigma_{ii}\sigma_{kk}+\frac{1+\alpha}{2}\sigma_{ik}(\sigma_{ik}+\sigma_{ki})\right.\\&\left.+\frac{1-\alpha}{2}\sigma_{ik}(\sigma_{ik}-\sigma_{ki})\right]Z+D,\end{aligned} \tag{5}$$

where summation over repeated indices is assumed. For simplicity, we set $\kappa_1=1$. To derive this equation we differentiated Eq. (4) with respect to t and expressed $\dot{u}_k^i$ using Eq. (1). We made use of the following identities:

$$\langle i\sigma_{ik}u^l_k u^i_l \exp(i\sigma_{mn}u^m_n)\rangle = -i\sigma_{ik}\frac{\partial^2}{\partial\sigma_{lk}\,\partial\sigma_{il}}Z, \tag{6}$$

$$\begin{aligned}\langle i\sigma_{ik}u^l u^i_{lk}\exp(i\sigma_{mn}u^m_n)\rangle &= \left\langle u^l\frac{\partial}{\partial x^l}\exp(i\sigma_{mn}u^m_n)\right\rangle \\ &= -\langle u^l_l \exp(i\sigma_{mn}u^m_n)\rangle \\ &= i\,\delta_{ik}\frac{\partial}{\partial\sigma_{ik}}Z,\end{aligned} \tag{7}$$

and

$$\langle i\sigma_{ik}f^i_k \exp(i\sigma_{mn}u^m_n)\rangle = \frac{1}{2}\sigma_{ik}\sigma_{mn}\kappa^{im}_{kn}(0)Z. \tag{8}$$

The D term in Eq. (5) describes the contribution of the dissipation and in steady state is given by

$$D = \lim_{\nu\to 0}\langle \nu i\sigma_{rs}\,\Delta u^r_s \exp(i\sigma_{kl}u^k_l)\rangle. \tag{9}$$

Without this term the steady state in Eq. (5) does not exist. This term cannot be closed without further assumptions. In [1], assumptions about scaling invariance, Galilean invariance, and the operator product expansion were applied to close the analogous term for the velocity-*difference* PDF. It is not obvious *a priori* that these methods can be applied to our problem, since the limits $y\to 0$ and $\nu\to 0$ may not be interchangeable.

Nevertheless, it was observed in the numerical simulations in [10] that the velocity-difference and the velocity-gradient PDF's coincide for the Galilean invariant region $\Delta u \ll u_{\rm rms}$ $[u_{\rm rms}=(\kappa_0 L)^{1/3}]$ in the one-dimensional case. This suggests that the velocity-difference PDF is contributed to by *smooth* parts of the velocity field in the Galilean invariant region, and therefore the limits $y\to 0$ and $\nu\to 0$ *are* interchangeable.

Another important result of [10] is that the β anomaly introduced in [1] is absent for regular forcing. We assume that this is true for the multidimensional case as well. Under this assumption the D term in the multidimensional case should be expanded as

$$D = aZ. \tag{10}$$

This is the only assumption we use in what follows. We refer the reader to Refs. [1–3,10] for more details and discussions on the underlying ideas. We will see that this assumption is self-consistent; the anomaly a can be found from the conditions of positivity, finiteness, and normalizability of the PDF. These conditions can be easily imposed on the PDF in u space. We therefore transform Eq. (5) to u space, using

$$P(\nabla u) = \int d\sigma\, Z(\sigma)e^{-i\sigma_{kl}u^k_l}, \tag{11}$$

where $d\sigma = \Pi_{i,k}\,d\sigma_{ik}$ is the measure in d^2-dimensional space of the elements of the matrix σ_{ik}. In the steady state, the equation takes the form

$$u^i_i P + \frac{\partial}{\partial u^i_l}(u^i_k u^k_l P) + \left[\alpha\frac{\partial^2}{\partial u^i_i\,\partial u^k_k} + \frac{1+\alpha}{2}\frac{\partial}{\partial u^i_k}\left(\frac{\partial}{\partial u^i_k}+\frac{\partial}{\partial u^k_i}\right) + \frac{1-\alpha}{2}\frac{\partial}{\partial u^i_k}\left(\frac{\partial}{\partial u^i_k}-\frac{\partial}{\partial u^k_i}\right)\right]P = -aP. \tag{12}$$

This is the general equation for the PDF. The force in this equation can have different symmetry properties, which correspond to different values of the parameter α in Eq. (3). No restrictions have so far been imposed on the velocity field either.

Equation (12) can be simplified for the *gradient* force $\mathbf{f} = \nabla\phi$. This choice corresponds to $\alpha=1$ and allows us to look for a solution in the factorized form

$$P = \left[\prod_{i<k}\delta(u^i_k - u^k_i)\right]\tilde{P}(\tilde{u}), \tag{13}$$

where $\tilde{u}$ denotes the symmetric part of the matrix u_{ik}. Physically, this means that we have restricted our consideration to *gradient* fluctuations of the velocity field, $\mathbf{u}=\nabla h$. We refer to this case as multidimensional Burgers turbulence. It has been considered by completely different methods in [5,7]. The spirit of our method is most close to the consideration of [8]. Equation (12) with ansatz (13) can be cast into the following form:

$$3\tilde{u}^i_i\tilde{P} + \tilde{u}^i_k\tilde{u}^k_l\frac{\partial}{\partial u^i_l}\tilde{P} + \left[\frac{\partial^2}{\partial u^i_i\,\partial u^k_k} + 2\frac{\partial^2}{\partial u^i_k\,\partial u^i_k}\right]\tilde{P} = -a\tilde{P}. \tag{14}$$

In what follows we will consider only the function $\tilde{P}$ and will omit the overtilde sign.

Equations (12) and (14) help to reveal the physical sense of the a anomaly. Integrating these equations with respect to u^i_k, one gets

$$\langle u^i_i\rangle = -a, \tag{15}$$

which means that this anomaly describes the average measure loss due to compressibility and presence of shocks. It can also be interpreted as the mean rate of density accumulation on shocks in the Lagrangian picture:

$$\dot{\rho}(y,t) + u^i_i\rho(y,t) + a\rho(y,t) = 0, \tag{16}$$

where y is the Lagrangian coordinate, u^i_i represents the smooth part of the velocity field, and $a=\langle u^i_i\rangle_{\rm shocks}$. We note an interesting analogy with a mean-field approximation: a is introduced as the mean field in the dynamical equations and then is found self-consistently from Eq. (14). The interpretation (16) is important, since it allows one to introduce the anomaly on the level of the stochastic Langevin equation.

In general, the PDF should depend only on invariants with respect to space rotations. For the d-dimensional space, there are exactly d such invariants, which can be chosen as the eigenvalues of the matrix $\tilde{u}_{ik}$. Let us denote them as $\lambda_1,\lambda_2,\ldots,\lambda_d$. Equation (14) can be rewritten for the function P depending on only these variables:

$$\sum_{k=1}^{d}\left(3\lambda_k+\lambda_k^2\frac{\partial}{\partial\lambda_k}\right)P+\left(\sum_{k=1}^{d}\frac{\partial}{\partial\lambda_k}\right)^2 P+2\sum_{k=1}^{d}\left(\frac{\partial}{\partial\lambda_k}\right)^2 P$$

$$+\sum_{i,k}\frac{1}{\lambda_i-\lambda_k}\left(\frac{\partial}{\partial\lambda_i}-\frac{\partial}{\partial\lambda_k}\right)P=-aP. \tag{17}$$

To derive this equation we used the following expression for the matrix Laplacian, known in the theory of matrix models [13,14]:

$$\nabla_{\tilde{u}}^2=\sum_i\frac{\partial^2}{\partial\lambda_i^2}+\frac{1}{2}\sum_{i,k}\frac{1}{\lambda_i-\lambda_k}\left(\frac{\partial}{\partial\lambda_i}-\frac{\partial}{\partial\lambda_k}\right). \tag{18}$$

Equation (17) has an infinite number of solutions. The physically reasonable solution should satisfy conditions of positivity, finiteness, and normalizability, exactly in the same manner as the ground state is determined in quantum mechanics. The solution should also be symmetrical with respect to the arguments $\lambda_1,\ldots,\lambda_d$. These conditions should determine the unknown parameter a. This parameter depends only on the symmetry properties of the external force and on the space dimensionality.

As in the one-dimensional case, the asymptotics of the solutions can be found by balancing different terms in Eq. (17). If we balance the advective and force terms, we will get the PDF tail in the region where the dissipative effects are negligible. In the one-dimensional case this corresponds to the right tail of the PDF. This tail decays hyperexponentially fast. In the multidimensional case the analogous asymptotic should have the form $P\propto\exp[S(\lambda_1,\ldots,\lambda_d)]$. The function S should be symmetric with respect to its arguments $\lambda_1,\ldots,\lambda_d$. The asymptotic can be simply found for large positive λ's in the direction close to $\lambda_1=\cdots=\lambda_d$:

$$P\propto\exp\left[\frac{-\Lambda^3}{3d^2(d+2)}\right], \tag{19}$$

where $\Lambda\equiv\mathrm{Tr}(\tilde{u}_{ik})=\lambda_1+\cdots+\lambda_d$. The same asymptotic for large λ can also be obtained by the instanton methods [7,8] applied directly to quantum mechanics (17).

The tail, corresponding to large negative λ's (the ''left'' tail), decays rather slowly. The explanation is simple. Burgers shocks always have negative velocity jumps, and therefore large positive velocity gradients are less probable than large negative ones. The left tail is determined by large negative gradients, and to obtain it we should neglect the force term in Eq. (17). We find

$$P\propto\frac{G(I_{ik})}{(\lambda_1\cdots\lambda_d)^3}, \tag{20}$$

where G is some function and $I_{ik}=(\lambda_i-\lambda_k)/\lambda_i\lambda_k$ are $d-1$ independent invariants of the characteristic equations for Eq. (17). The finite solution, which is nonvanishing for $\lambda_1=\cdots=\lambda_d$, has the form

$$P\propto\frac{1}{(\lambda_1\cdots\lambda_d)^3}\equiv\mathrm{Det}^{-3}(\tilde{u}_{ik}). \tag{21}$$

The function obtained from Eq. (17) should be normalized with respect to the flat measure in the $[d(d+1)/2]$-dimensional space of elements of the symmetric matrix $\tilde{u}_{ik}$. In λ space, this normalization is performed as follows:

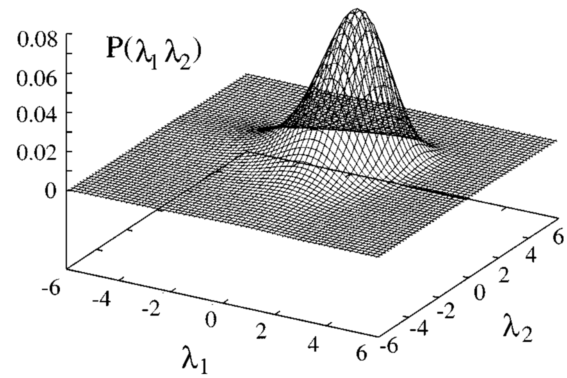

FIG. 1. Velocity-gradient probability-density function $P(\lambda_1,\lambda_2)$.

$$\int|\Delta(\lambda)|P(\lambda)\prod_{k=1}^{d}d\lambda_k=1, \tag{22}$$

where $\Delta(\lambda)=\Pi_{i<j}(\lambda_i-\lambda_j)$ is the Van der Monde determinant; for details, see [13–15].

III. NUMERICAL SOLUTION FOR THE TWO-DIMENSIONAL CASE

In this section we solve Eq. (17) numerically in the two-dimensional case. The purpose of these calculations is to show that Eq. (5) with the anomaly term (10) does have a steady state, at least for the gradient velocity field.

We have used the relaxation method and started with some arbitrary but *symmetrical* initial distribution. The numerical value for the anomaly turned out to be $a=1.30\pm0.02$. The PDF has hyperexponential and powerlike tails and is presented in Fig. 1. The PDF is normalized according to Eq. (22). Plotted on the horizontal axes are λ_1 and λ_2.

Figure 2 shows the same PDF for the diagonal direction $\lambda_1=\cdots=\lambda_d$. The left tail decays as $1/\Lambda^6$; the right tail

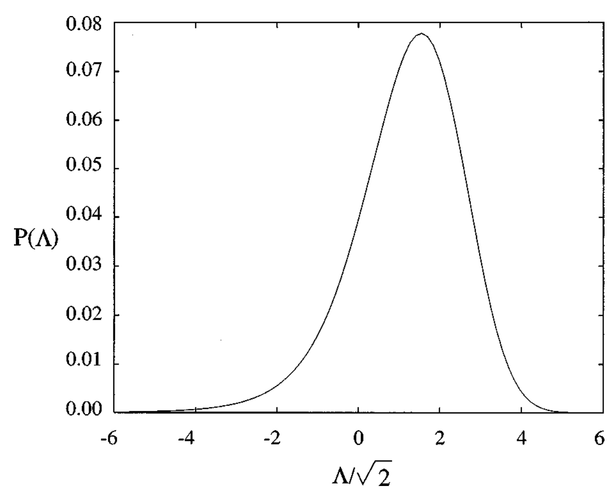

FIG. 2. Velocity-gradient PDF $P(\Lambda)$ for the diagonal direction $\lambda_1=\cdots=\lambda_d$.

asymptotic is $P \propto \exp(-\Lambda^3/48)$, in agreement with Eqs. (19) and (21). Here $P(\Lambda)$ is plotted vs $\sqrt{\lambda_1^2+\lambda_2^2}=\Lambda/\sqrt{2}$.

IV. CONCLUDING REMARKS

The crucial assumption in our treatment of the dissipative anomaly is the assumption that only smooth parts of the velocity field contribute to the anomaly term. We can exploit this assumption further to generalize our results to the velocity-*difference* PDF. After the velocity-gradient Z function (4) is found, the velocity-difference Z function can be constructed as follows:

$$Z_v(\zeta_i, y^k) = Z(\zeta_i y^k) \equiv \langle \exp(i\zeta_i y^k u_k^i)\rangle; \tag{23}$$

i.e., we simply changed $\sigma_{ik} \rightarrow \zeta_i y_k$ in Eq. (4). The Fourier transform with respect to ζ will then give the velocity-difference PDF.

Analogously, one can obtain a PDF for $\nabla \cdot$u. For this purpose one should set $\sigma_{ik} \rightarrow \delta_{ik}\zeta$. Such a PDF was investigated numerically in [16], though the Reynolds number was not large enough to obtain the inertial range.

Finally, we would like to note that the absence of the β anomaly, which we assumed in our consideration, cannot be a universal fact. It was conjectured in [3] that different dissipative regularizations [e.g., hyperdissipation $(-1)^{p}\partial^{2p}/\partial x^{2p}$] can lead to different steady states. This assumption is natural for the language of the OPE: different dissipative operators should have different expansion coefficients a and b (we use the notation of [1]). Moreover, some analog of the β anomaly can also be present in Eq. (12), since it describes a general velocity field, without "gradient" restriction (13).

These questions are under consideration. The results will be reported elsewhere.

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